

An improvement upon unmixed decomposition of an algebraic variety *

Zhenyi Ji[†], Yongbin Li[‡]

*School of Applied Mathematics,
University of Electronic Science and Technology of China,
Chengdu, Sichuan, 610054, China*

Abstract

Decomposing an algebraic variety into irreducible or equidimensional components is a fundamental task in classical algebraic geometry and has various applications in modern geometry engineering. Several researchers studied the problem and developed efficient algorithms using *Gröbner* basis method. In this paper, we try to modify the computation of unmixed decomposition of an algebraic variety based on improving the computation of $Zero(sat(\mathbb{T}))$, where \mathbb{T} is a triangular set in $\mathbf{K}[\mathbf{X}]$.

Keywords: unmixed decomposition, weakly non-degenerate conditions, *Wu's* characteristic set, U-set.

1 Introduction

Let \mathbf{K} be a field of characteristic 0 and $\mathbf{K}[x_1, x_2, \dots, x_n]$ (or $\mathbf{K}[\mathbf{X}]$ for short) the ring of polynomials in the variables (x_1, x_2, \dots, x_n) with coefficients in \mathbf{K} . A polynomial set is a finite set \mathbb{P} of nonzero polynomials in $\mathbf{K}[\mathbf{X}]$. The ideal of $\mathbf{K}[\mathbf{X}]$ generated by all elements of \mathbb{P} is denoted by $Ideal(\mathbb{P})$ and the algebraic variety of \mathbb{P} is denoted by $Zero(\mathbb{P})$. The method of *Gröbner* bases introduced by *Buchberger*^[1,2] provides a powerful device for computing a basis of $Ideal(\mathbb{P})$. It is well known that *Wu*^[22] provided an efficient method for constructing a *Wu's* characteristic set of every polynomial set to compute the variety of \mathbb{P} in 1978. Therefrom, various algorithms for computing triangular decomposition of polynomial sets and *systems* are developed by some researchers^[3,4,10,11,17,19,23,24,25].

*Partially supported by a NKBRPC (2004CB318000)

[†]E-mail: jizhenyi0010@163.com (Z.Y.Ji)

[‡]E-mail: yongbinli@uestc.edu.cn (Y.B.Li)

Based on various triangular decompositions for polynomial systems, including the famous *Wu's* characteristic set method, and the *Gröbner* Basis method we can get the unmixed decomposition of an algebraic variety.

According to the analytic method established by *Zhang et.al*^[28], we get the modified method to compute characteristic series. Furthermore, we try to improve the computation of $Zero(sat(\mathbb{T}))$, where \mathbb{T} is a triangular set. Some examples can illustrate our improvement.

2 Preliminaries

2.1 *Wu's* characteristic set

For any polynomial $p \notin \mathbf{K}$, the biggest index k such that $deg(p, x_k) > 0$ is called the *class*, x_k the *leading variable*, $deg(p, x_k)$ the *leading degree* of p , and $lcoef(p, x_k)$ the *leading coefficient* of p , denoted by $cls(p)$, $lv(p)$, $ldeg(p)$, $ini(p)$, respectively.

Definition 2.1.1. A finite nonempty ordered set $\mathbb{T} = [f_1, f_2, \dots, f_s]$ of polynomials in $\mathbf{K}[\mathbf{X}] \setminus \mathbf{K}$ is called a *triangular set* if $cls(f_1) < cls(f_2) < \dots < cls(f_s)$. Triangular set \mathbb{T} is written as the following form

$$\mathbb{T} = [f_1(u_1, \dots, u_r, y_1), \dots, f_s(u_1, \dots, u_r, y_1, \dots, y_s)] \quad (1)$$

where $(u_1, \dots, u_r, y_1, \dots, y_s)$ is a permutation of (x_1, \dots, x_n) .

Let $f \neq 0$ be a polynomial and g any polynomial in $\mathbf{K}[\mathbf{X}]$, the *pseudo-remainder* of g with respect to f in $lv(f)$ is denoted by $prem(g, f, lv(f))$. One can find a formal definition of *pseudo-remainder*^[7,18] or two alternative ones^[18,19]. For any polynomial p and triangular set \mathbb{T} $prem(p, \mathbb{T})$ stands for the *pseudo-remainder* of p with respect to \mathbb{T} is defined by

$$prem(p, \mathbb{T}) = prem(\dots prem(p, f_s, y_s), \dots, f_1, y_1). \quad (2)$$

One can easily deduce the following *pseudo-remainder formula*

$$\prod_{i=1}^s ini(f_i)^{d_i} p = \sum_{i=1}^s q_i f_i + prem(p, \mathbb{T}), \quad (3)$$

where each d_i is a nonnegative integer and $q_i \in \mathbf{K}[\mathbf{X}]$ for $1 \leq i \leq s$.

For any polynomial set $\mathbb{P} \subset \mathbf{K}[\mathbf{X}]$, we write

$$prem(\mathbb{P}, \mathbb{T}) \triangleq \{prem(p, \mathbb{T}) | p \in \mathbb{P}\}.$$

Given two polynomials $f, g \in \mathbf{K}[\mathbf{X}]$, the *Sylvester resultant* of f and g with respect to some x_k ($1 \leq k \leq n$) is denoted by $\text{res}(f, g, x_k)$. Let p be any polynomial and $\mathbb{T} = [f_1, f_2, \dots, f_s]$ a triangular set in $\mathbf{K}[\mathbf{X}]$ as (1). The polynomial

$$\text{res}(p, \mathbb{T}) \triangleq \text{res}(\dots \text{res}(p, f_s, y_s), \dots, f_1, y_1)$$

is called the *resultant* of p with respect to \mathbb{T} .

Let \mathbb{T} is a triangular set as (1) and p any polynomial, p is said to be reduced with respect to \mathbb{T} if $\deg(p, y_i) < \deg(f_i, y_i)$ for all i . \mathbb{T} is said to be *noncontradictory ascending set* if every $f \in \mathbb{T} \cup (\text{ini}(\mathbb{T}))$ is reduced to $\mathbb{T} \setminus \{f\}$.

Definition 2.1.2. A triangular set $\mathbb{T} = [f_1, f_2, \dots, f_s]$ is called perfect, if $\text{Zero}(\mathbb{T}, \text{ini}(\mathbb{T})) \neq \emptyset$.

Definition 2.1.3.^[22] A noncontradictory ascending set \mathbb{T} is called a *Wu's characteristic set* of polynomial set $\mathbb{P} \subset \mathbf{K}[\mathbf{X}]$ if

$$\mathbb{T} \subset \text{Ideal}(\mathbb{P}), \quad \text{prem}(\mathbb{P}, \mathbb{T}) = \{0\}.$$

Definition 2.1.4. A finite set $\mathbb{T}_1, \mathbb{T}_2, \dots, \mathbb{T}_s$ is called a *characteristic series* of polynomial set \mathbb{P} in $\mathbf{K}[\mathbf{X}]$ if the following zero decomposition holds

$$\text{Zero}(\mathbb{P}) = \bigcup_{i=1}^s \text{Zero}(\mathbb{T}_i / \text{ini}(\mathbb{T}_i)) \quad (4)$$

and $\text{prem}(\mathbb{P}, \mathbb{T}_i) = \{0\}$ for every i .

Definition 2.1.5.^[10,25] A triangular set $\mathbb{T} = [f_1, f_2, \dots, f_s]$ is called a *regular set* if $\text{res}(I, \mathbb{T}) \neq 0$ for all $I \in \text{ini}(\mathbb{T})$.

Definition 2.1.6.^[4,19] A triangular set $\mathbb{T} = [f_1, f_2, \dots, f_s]$ as (1) is called a *normal set* if $\text{ini}(\mathbb{T}) \in \mathbf{K}[\mathbf{U}]$.

Definition 2.1.7. Let \mathbb{T} be a triangular set in $\mathbf{K}[\mathbf{X}]$. The *saturation ideal* of \mathbb{T}

$$\text{sat}(\mathbb{T}) \triangleq \text{Ideal}(\mathbb{T}) : J^\infty = \{g \in \mathbf{K}[\mathbf{X}] \mid J^q g \in \text{Ideal}(\mathbb{T}) \text{ for some } q > 0\},$$

where $J = \prod_{f \in \mathbb{T}} \text{ini}(f)$.

One can compute a basis of $\text{sat}(\mathbb{T})$ by the following Lemma.

Lemma 2.1.8.^[5,8,20] Let $\mathbb{T} = [f_1, f_2, \dots, f_s]$ be a triangular set in $\mathbf{K}[\mathbf{X}]$, z is a new variable, $\mathbb{H} = \mathbb{T} \cup \{zJ - 1\} = \{f_1, f_2, \dots, f_s, zJ - 1\}$, Gb be the *Gröbner*

basis of \mathbb{H} with respect to a *lexicographic ordering* where z is greater than every x_i . Then

$$\text{sat}(\mathbb{T}) = \text{Ideal}(\mathbb{H}) \cap \mathbf{K}[\mathbf{X}] = \text{Ideal}(Gb \cap \mathbf{K}[\mathbf{X}]). \quad (5)$$

2.2 The theory of weakly non-degenerate conditions and its application

Let \mathbb{T} be as (1), we denote \mathbb{C}_{f_i} the set of all the nonzero coefficients of f_i in y_i , $\mathbb{R}_{f_i} = \{\text{res}(c, \mathbb{T}) \neq 0 : c \in \mathbb{C}_{f_i}\}$ for any $f_i \in \mathbb{T}$. For any $\bar{\mathbf{z}} = (\bar{\mathbf{u}}, \bar{y}_1, \dots, \bar{y}_s) \in \text{Zero}(\mathbb{T})$, we write $\bar{\mathbf{z}}^{[j]}$ for $\bar{\mathbf{u}}, \bar{y}_1, \dots, \bar{y}_j$ or $(\bar{\mathbf{u}}, \bar{y}_1, \dots, \bar{y}_j)$ with $0 \leq j \leq s$.

Definiton 2.2.1.^[28] Let \mathbb{T} as (1) be a regular set in $\mathbf{K}[\mathbf{X}]$. A zero $\mathbf{z}_0 \in \text{Zero}(\mathbb{T})$ is called a *quasi-normal zero* if $\mathbf{z}_0^{\{i-1\}} \notin \text{Zero}(\mathbb{C}_{f_i})$ for any $1 \leq i \leq s$, also said to satisfying the *weakly non-degenerate conditions*.

The following definition is an extension of the concept of *quasi-normal zero* of regular set to triangular set.

Definiton 2.2.2.^[13] Let $\mathbb{T} = [f_1, f_2, \dots, f_s]$ as (1) be a triangular set in $\mathbf{K}[\mathbf{X}]$. A zero $\mathbf{z}_0 \in \text{Zero}(\mathbb{T})$ is called a *quasi-normal zero* of \mathbb{T} if for any $1 \leq i \leq s$, *either conditions holds*:

- a. $I_i(\mathbf{z}_0^{\{i-1\}}) \neq 0$ if $\text{res}(I_i, \mathbb{T}) = 0$;
- b. $\mathbf{z}_0^{\{i-1\}} \notin \text{Zero}(\mathbb{C}_{f_i})$ if $\text{res}(I_i, \mathbb{T}) \neq 0$.

For any triangular set \mathbb{T} in $\mathbf{K}[\mathbf{X}]$, we denote $QnZero(\mathbb{T})$ the set of all *quasi-normal zeros* of \mathbb{T} and $\overline{QnZero(\mathbb{T})}^E$ the closure of $QnZero(\mathbb{T})$ in topological space \mathbf{K}^n , where \mathbf{K}^n is induced by follow metric

$$|\mathbf{z} - \mathbf{z}^*| = \max\{|x_1 - x_1^*|, |x_2 - x_2^*|, \dots, |x_n - x_n^*|\} \text{ for any } \mathbf{z}, \mathbf{z}^* \in \tilde{\mathbf{K}}^n,$$

then we have the following theorem.

Theorem 2.2.3.^[13] For any triangular set $\mathbb{T} = [f_1(\mathbf{u}, y_1), \dots, f_s(\mathbf{u}, y_1, \dots, y_s)]$, we have

$$\text{Zero}(\mathbb{T}/\text{ini}(\mathbb{T})) \subseteq \overline{QnZero(\mathbb{T})}^E \subseteq \text{Zero}(\text{sat}(\mathbb{T})).$$

The following definition plays an important role in this paper.

Definition 2.2.4.^[9,13] Let \mathbb{T} be a triangular set in $\mathbf{K}[\mathbf{X}]$, We establish $\mathbb{U}_{\mathbb{T}} \triangleq \{c \in \mathbb{C}_f : \text{res}(\text{ini}(f), \mathbb{T}) \neq 0 \text{ and } \mathbb{R}_f \cap K = \emptyset, f \in \mathbb{T}\} \cup$

$$\{c : \text{res}(c, \mathbb{T}) = 0, c \in \text{ini}(\mathbb{T})\}.$$

Remark: This definition has slightly difference with the notion in^[9,13].

Example 2.2.5. Let $\mathbb{T} = [f_1, f_2, f_3]$, under $x_1 \prec x_2 \prec x_3 \prec x_4$, where

$$\begin{aligned} f_1 &= x_1 x_2^2 + x_2 + 2x_1^2, \\ f_2 &= x_2 x_3 + x_1 x_2^2 + x_2 x_1 + 2x_1, \\ f_3 &= x_2 x_4^2 - x_4 - x_2 - x_3. \end{aligned}$$

By above notation, we know

$$\begin{aligned} \mathbb{C}_{f_1} &= \{x_1, 1, 2x_1^2\}, \\ \mathbb{C}_{f_2} &= \{x_2, x_1 x_2^2 + x_1 x_2 + 2x_1\}, \\ \mathbb{C}_{f_3} &= \{x_3, -1, -x_2 - x_3\}. \end{aligned}$$

It is easy to see that

$$\begin{aligned} \mathbb{R}_{f_1} &= \mathbb{C}_{f_1} = \{x_1, 1, 2x_1^2\}, \\ \mathbb{R}_{f_2} &= \{2x_1^2, x_1^2(4x_1^4 - 6x_1^3 + 2x_1^2 - 2x_1 + 2)\}, \\ \mathbb{R}_{f_3} &= \{2x_1, -1, 4x_1^6 - 14x_1^5 + 14x_1^4 + 2x_1^2 - 2x_1\}. \end{aligned}$$

Thus $\mathbb{U}_{\mathbb{T}} = \{x_2\}$.

Similarly, one can compute that $\mathbb{U}_{\mathbb{T}^*} = \emptyset$ where $\mathbb{T}^* = [g_1, g_2, g_3]$

$$\begin{aligned} g_1 &= -x_2 x_3^2 - x_3 + x_1 x_2^2 - x_2 x_1, \\ g_2 &= x_1 x_4^2 + x_3 x_4^2 + x_4 + x_3 - 2x_2 + x_2 x_1 + x_1 x_2^2, \\ g_3 &= x_3 x_5^2 + 2x_5 + 2x_1 x_2^2 + x_2 x_1 + 2x_3, \end{aligned}$$

under $x_1 \prec x_2 \prec x_3 \prec x_4 \prec x_5$.

Theorem 2.2.6.^[13] For any triangular set \mathbb{T} , we have

$$\text{Zero}(\mathbb{T}/\mathbb{U}_{\mathbb{T}}) \subseteq \overline{\text{QnZero}(\mathbb{T})}^E \subseteq \text{Zero}(\text{sat}(\mathbb{T})).$$

Corollary 2.2.7.^[13] Let \mathbb{T} be a triangular set in $\mathbf{K}[\mathbf{X}]$ with $\mathbb{U}_{\mathbb{T}} = \emptyset$. Then

$$\text{Zero}(\mathbb{T}) = \text{Zero}(\text{sat}(\mathbb{T})).$$

Based on the the theory of weakly non-degenerate conditions we get the following modified algorithm *CharserA*^[9,13] to compute characteristic series.

Algorithm CharserA: $\Psi \leftarrow \text{CharserA}(\mathbb{P})$. Given a nonempty polynomial set \mathbb{P} in $\mathbf{K}[\mathbf{X}]$, this algorithm computes a finite sets Ψ such that

$$\text{Zero}(\mathbb{P}) = \bigcup_{\mathbb{T} \in \Psi} \text{Zero}(\mathbb{T}/\mathbb{U}_{\mathbb{T}})$$

C1 : Set $\Phi \leftarrow \{\mathbb{P}\}$, $\Psi \leftarrow \emptyset$.

C2 : While $\Phi \neq \emptyset$ do:

C2.1. Let \mathbb{F} be an element of Ψ and set $\Psi \leftarrow \Psi \setminus \mathbb{F}$.

C2.2 Compute $\mathbb{T} \leftarrow \text{Charset}(\mathbb{F})$.

C2.3 If \mathbb{T} is noncontradictory, then compute $\mathbb{U}_{\mathbb{T}}$.

C2.4 If $\mathbb{U}_{\mathbb{T}} = \emptyset$, then set $\Psi \leftarrow \Psi \cup \mathbb{T}$.

C2.5 If $\mathbb{U}_{\mathbb{T}} \neq \emptyset$, then set

$$\Psi \leftarrow \Psi \cup \mathbb{T}, \Phi \leftarrow \Phi \cup \{\mathbb{F} \cup \mathbb{T} \cup \{I\} : I \in \mathbb{U}_{\mathbb{T}}\}.$$

Example 2.2.8. Let $\mathbb{P} = \{p_1, p_2, p_3\}$ in $\mathbf{K}[x_1, x_2, x_3, x_4]$, where

$$\begin{aligned} p_1 &= x_3^2 + 2x_3x_2 + x_1x_2 + 2, \\ p_2 &= x_1^3 + 2 - 4x_3x_2^2 - 2x_2x_3^2 + 2x_1^2 - 2x_2, \\ p_3 &= x_2x_3x_4^2 + x_4 + x_1^2 + 2x_3 + x_2. \end{aligned}$$

Under the variable ordering $x_1 \prec x_2 \prec x_3 \prec x_4$. By the above description, one can easily get $CharserA = \{\mathbb{T}\}$, where

$$\mathbb{T} = [2x_1x_2^2 + 2x_2 + x_1^3 + 2x_1^2 + 2, x_3^2 + 2x_2x_3 + x_1x_2 + 2, x_2x_3x_4^2 + x_4 + x_1^2 + 2x_3 + x_2].$$

It is easy to see that $\mathbb{U}_T = \emptyset$, this implies that

$$Zero(\mathbb{P}) = Zero(\mathbb{T})$$

Compared with *MMP*, *epsilon*, *Regular*, we get the following zero decomposition directly.

$$Zero(\mathbb{P}) = Zero(\mathbb{T}_1/\{x_1, x_2x_3\}) \bigcup_{i=2}^5 Zero(\mathbb{T}_i). \text{ (MMP)}$$

where

$$\mathbb{T}_1 = [2x_1x_2^2 + 2x_2 + 2 + 2x_1^2 + x_1^3, x_3^2 + 2x_2x_3 + x_1x_2 + 2, x_2x_3x_4^2 + x_4 + x_1^2 + 2x_3 + x_2].$$

$$\mathbb{T}_2 = [-x_1^7 - 4x_1^6 - 4x_1^5 - 4x_1^4 - 12x_1^3 - 8x_1^2 - 4x_1 - 8, -2x_2 + x_1^3 + 2x_1^2 + 2, -x_1^3x_3 - 2x_1^2x_3 - 2x_3, x_1^4x_4 + 2x_1^3x_4 + 2x_1x_4 + x_1^6 + 2x_1^5 - 4x_1^2 - 4].$$

$$\mathbb{T}_3 = [-2x_1^3 - 4x_1^2 - 4, 4x_2, 4x_3^2 + 8, 4x_4 + 8x_3 + 4x_1^2].$$

$$\mathbb{T}_4 = [-x_1^3 - 2x_1^2 - 2, -2x_2, -2x_3^2 - 4, -2x_4 - 4x_3 - 2x_1^2].$$

$$\mathbb{T}_5 = [2x_1, 4x_2 + 4, 2x_3^2 - 4x_3 + 4, -4x_3x_4^2 + 4x_4 + 8x_3 - 4].$$

$$Zero(\mathbb{P}) = Zero(\mathbb{T}_1/\{x_1, x_2x_3\}) \bigcup_{i=2}^4 Zero(\mathbb{T}_i). \text{ (epsilon)}$$

where

$$\mathbb{T}_1 = [2x_1x_2^2 + 2x_2 + 2 + 2x_1^2 + x_1^3, x_3^2 + 2x_2x_3 + x_1x_2 + 2, x_2x_3x_4^2 + x_4 + x_1^2 + 2x_3 + x_2].$$

$$\mathbb{T}_2 = [x_1^4 + 2x_1^3 + 2x_1 + 4, 2x_2 - x_1^3 - 2 - 2x_1^2, x_3, 2x_4 + 4x_1^2 + x_1^3 + 2].$$

$$\mathbb{T}_3 = [x_1, 1 + x_2, x_3^2 + 2 - 2x_3, x_3x_4^2 - x_4 - 2x_3 + 1].$$

$$\mathbb{T}_4 = [x_1^3 + 2 + 2x_1^2, x_2, x_3^2 + 2, x_4 + x_1^2 + 2x_3].$$

$$Zero(\mathbb{T}) = Zero(\mathbb{T}/\{x_1, x_2x_3\}). \text{ (Regular)}$$

where

$$\mathbb{T} = [2x_1x_2^2 + 2x_2 + 2 + 2x_1^2 + x_1^3, x_3^2 + 2x_2x_3 + x_1x_2 + 2, x_2x_3x_4^2 + x_4 + x_1^2 + 2x_3 + x_2];$$

Other experiment comparison see table 1.

Table 1: Number of triangular sets for nine test examples.

Polynomial set	$MMP^{[7]}$	$epsilon^{[21]}$	$Regular^{[16]}$	$CharserA^{[9,13]}$
	<i>Branches</i>	<i>Branches</i>	<i>Branches</i>	<i>Branches</i>
\mathbb{P}_1	1	1	2	1
\mathbb{P}_2	2	3	5	2
\mathbb{P}_3	3	3	6	1
\mathbb{P}_4	3	2	1	2
\mathbb{P}_5	4	3	1	1
\mathbb{P}_6	6	?	2	4
\mathbb{P}_7	9	5	7	1
\mathbb{P}_8	9	6	1	2
\mathbb{P}_9	6	4	4	3

3 Traditional method for unmixed decomposition

An algebraic variety is said to be unmixed or equidimensional if all irredundant irreducible components have the same dimension.

Refer to the zero decomposition (4) which provides a representation of the variety $Zero(\mathbb{P})$ in terms of its subvarieties determined by \mathbb{T}_i . However, each $Zero(\mathbb{T}_i/ini(\mathbb{T}_i))$ is not necessarily an algebraic variety, it is a *quasi-algebraic variety*. In what follows, we shall see how a corresponding variety decomposition may be obtained by determining, from each \mathbb{T}_i , a finite set of polynomials.

Theorem 3.1 Let \mathbb{P} be a polynomial set in $\mathbf{K}[\mathbf{X}]$, $\mathbb{T}_1, \mathbb{T}_2, \dots, \mathbb{T}_s$ is a characteristic series of \mathbb{P} , then we have

$$Zero(\mathbb{P}) = \bigcup_{i=1}^s Zero(sat(\mathbb{T}_i)). \quad (6)$$

The following result provides a useful criterion for removing some redundant subvarieties in the decomposition (4) without computing their defining sets.

Lemma 3.2^[3] Let \mathbb{P} and \mathbb{T}_i be as in Theorem 3.1, if $|\mathbb{T}_i| > |\mathbb{P}|$, then we have

$$Zero(sat(\mathbb{T}_i)) \subset \bigcup_{\substack{1 \leq i \leq s \\ i \neq j}} Zero(sat(\mathbb{T}_j)). \quad (7)$$

The next theorem can make sure this decomposition is unmixed.

Theorem 3.3^[6] Let $\mathbb{T} = [f_1, f_2, \dots, f_s]$ be a triangular set in $\mathbf{K}[\mathbf{X}]$, if \mathbb{T} is not perfect, then $\text{sat}(\mathbb{T}) = \mathbf{K}[\mathbf{X}]$. If \mathbb{T} is perfect, then $\text{Zero}(\text{sat}(\mathbb{T}))$ is an unmixed variety of dimension $n - |\mathbb{T}|$.

For every i let \mathbb{G}_i be the finite basis of $\text{sat}(\mathbb{T}_i)$, which can be computed by Lemma 2.1.6. If $\text{sat}(\mathbb{T}_i) = \mathbf{K}[\mathbf{X}]$ then $\text{Zero}(\text{sat}(\mathbb{T}_i)) = \emptyset$, hence, we can remove it. Thus a variety decomposition of the following form is obtained:

$$\text{Zero}(\mathbb{P}) = \bigcup_{i=1}^s \text{Zero}(\mathbb{G}_i) \quad (8)$$

By theorem 3.3, each \mathbb{G}_i defines an unmixed algebraic variety.

4 Improvement

Based upon the theory of weakly non-degenerate conditions, we have the following result.

Theorem 4.1 Let $\mathbb{T} = [f_1, f_2, \dots, f_s]$ as (1) be a triangular set in $\mathbf{K}[\mathbf{X}]$, then

$$\text{Zero}(\text{sat}(\mathbb{T})) = \text{Zero}(\text{Ideal}(\mathbb{T}) : U^\infty) \quad (9)$$

where $U = \prod_{u \in \mathbb{U}_{\mathbb{T}}} u$.

Proof: It is easy to see that $\text{Zero}(\text{sat}(\mathbb{T})) \subset \text{Zero}(\text{Ideal}(\mathbb{T}) : U^\infty)$.

Next, to establish containment in the opposite direction, let

$$\mathbf{X} \in \text{Zero}(\text{Ideal}(\mathbb{T}) : U^\infty).$$

Equivalently,

$$\text{if } fU^m \in \text{Ideal}(\mathbb{T}) \text{ for some } m > 0, \text{ then } f(\mathbf{X}) = 0.$$

Now, let $f \in \text{Ideal}(\text{Zero}(\mathbb{T}/U))$, then fU vanishes on $\text{Zero}(\mathbb{T})$. Thus, by the *Nullstellensatz*, $fU \in \sqrt{\text{Ideal}(\mathbb{T})}$, so $(fU)^l \in \text{Ideal}(\mathbb{T})$ for some $l > 0$. Hence, $f^l \in \text{Ideal}(\mathbb{T}) : U^\infty$. we have $f(\mathbf{X}) = 0$. Thus

$$\mathbf{X} \in \text{Zero}(\text{Ideal}(\text{Zero}(\mathbb{T}/U))).$$

This establish that

$$\text{Zero}(\text{Ideal}(\mathbb{T}) : U^\infty) \subset \text{Zero}(\text{Ideal}(\text{Zero}(\mathbb{T}/U))).$$

We also know

$$\text{Zero}(\text{Ideal}(\mathbb{T}) : U^\infty) \supset \text{Zero}(\text{Ideal}(\text{Zero}(\mathbb{T}/U))),$$

then

$$\begin{aligned} \text{Zero}(\text{Ideal}(\mathbb{T}) : U^\infty) &= \overline{\text{Zero}(\text{Ideal}(\text{Zero}(\mathbb{T}/U)))} \\ &= \overline{\text{Zero}(\mathbb{T}/U)}. \end{aligned}$$

where $\overline{\text{Zero}(\mathbb{T}/U)}$ denotes the *Zariski closure* of $\text{Zero}(\mathbb{T}/U)$.

From the definition of *Zariski closure* and $Zero(\mathbb{T}/U) \subset Zero(sat(\mathbb{T}))$, then $Zero(sat(T)) \supset Zero(Ideal(\mathbb{T}) : U^\infty)$.

This completes the proof.

Applying $CharserA^{[9,13]}$ and above theorem we get the modified algorithm for unmixed decomposition of an algebraic variety.

Algorithm 4.2: $\Psi \leftarrow UnmVarDec(\mathbb{P})$. Given a nonempty set \mathbb{P} , this algorithm computes finite set Ψ of polynomial sets $\mathbb{G}_1, \dots, \mathbb{G}_s$ such that the decomposition (8) holds and each \mathbb{G}_i defines an unmixed algebraic variety.

U1: Compute $\Phi \leftarrow CharserA(\mathbb{P})$, and set $\Psi \leftarrow \emptyset$:

U2: While $\Phi \neq \emptyset$, do:

U2.1: Let \mathbb{T} be an element in Φ , and $\Phi \leftarrow \Phi \setminus \{\mathbb{T}\}$. if $|\mathbb{T}| > |\mathbb{P}|$, then go to U2:

U2.2: Compute *Gröbner* basis \mathbb{G} of $Ideal(\mathbb{T}) : U^\infty$ according to Lemma 2.1.8, and set $\Psi \leftarrow \Psi \cup \{\mathbb{G}\}$:

U3: While $\exists \mathbb{G}, \mathbb{G}^*$ such that $rem(\mathbb{G}, \mathbb{G}^*) = \{0\}$, do:
set $\Psi \leftarrow \Psi \setminus \{\mathbb{G}^*\}$.

where $rem(\mathbb{G}, \mathbb{G}^*) \triangleq \{rem(p, \mathbb{G}^*) | p \in \mathbb{G}\}$ and $rem(p, \mathbb{G}^*)$ see^[1,2] for details.

Example 4.3. Let $\mathbb{P} = \{f_1, f_2, f_3, f_4\}$ be a polynomial set in $\mathbf{K}[x_1, x_2, x_3, x_4]$ where

$$\begin{aligned} g_1 &= 2x_3^2x_1 + x_3^2x_2 + x_3 + x_1, \\ g_2 &= -x_2x_3x_4 + 2x_1x_2^2 - x_3x_4^2 + x_2 + 2x_1 - x_4, \\ g_3 &= x_5^2x_1^2 - x_2x_5^2 + x_5 - x_3x_4 + x_2 + x_1, \\ g_4 &= 2x_2^3x_3^2 + 2x_2x_3^3x_4 + 2x_4^2x_3^3 + 2x_2^2x_3 + x_2x_3x_4 + 2x_4x_3^2 + x_3x_4^2 - x_2 + \\ &\quad 2x_3 + x_4. \end{aligned}$$

Under variable ordering $x_1 \prec x_2 \prec x_3 \prec x_4 \prec x_5$ \mathbb{P} is decomposed into six characteristic sets \mathbb{T}_i^* such that

$$Zero(\mathbb{P}) = \bigcup_{i=1}^6 Zero(\mathbb{T}_i^*/ini(\mathbb{T}_i^*))$$

where

$$\begin{aligned} \mathbb{T}_1^* &= [2x_1x_3^2 + x_2x_3^2 + x_3 + x_1, -x_3x_4^2 - x_2x_3x_4 - x_4 + x_2 + 2x_1 + 2x_1x_2^2, -x_1^2x_5^2 \\ &\quad + x_2x_5^2 - x_5 + x_3x_4 - x_2 - x_1], \\ \mathbb{T}_2^* &= [x_2 - x_1^2, 2x_1x_3^2 + x_1^2x_3^2 + x_3 + x_1, x_3x_4^2 + x_4 + x_3x_1^2x_4 - 2x_1 - x_1^2 - \\ &\quad 2x_1^5, -x_5 + x_3x_4 - x_1 - x_1^2], \\ \mathbb{T}_3^* &= [x_2 + 2x_1, x_3 + x_1, x_1x_4^2 - x_4 - 2x_1^2x_4 + 8x_1^3, x_1^2x_5^2 + 2x_1x_5^2 + x_5 - \\ &\quad x_1 + x_1x_4], \\ \mathbb{T}_4^* &= [x_1, x_3, -x_4 + x_2, x_2x_5^2 - x_5 - x_2], \end{aligned}$$

$$\mathbb{T}_5^* = [x_1 + 2, x_2 - 4, x_3 - 2, 2x_4^2 + 9x_4 + 64, x_5 + 2 - 2x_4],$$

$$\mathbb{T}_6^* = [x_1, x_2, x_3, x_4, x_5].$$

\mathbb{T}_5^* and \mathbb{T}_6^* contains five polynomials and thus need not be considered for the variety decomposition by Lemma 3.2. In order to obtain an unmixed decomposition of $\text{Zero}(\mathbb{P})$, It remains to determine $\text{sat}(\mathbb{T}_1^*)$, $\text{sat}(\mathbb{T}_2^*)$, $\text{sat}(\mathbb{T}_3^*)$, $\text{sat}(\mathbb{T}_4^*)$ by computing the respectively *Gröbner* basis $\mathbb{G}_1, \mathbb{G}_2, \mathbb{G}_3, \mathbb{G}_4$ of $\mathbb{T}_1^* \cup \{1 + zx_3(2x_1 + x_2)(x_2 - x_1^2)\}$, $\mathbb{T}_2^* \cup \{1 - zx_3(2x_1 + x_1^2)\}$, $\mathbb{T}_3^* \cup \{1 - zx_1(x_1 + 2)\}$, $\mathbb{T}_4^* \cup \{1 - zx_2\}$ according to Lemma 2.1.8. The *Gröbner* base may be found to consist of 15, 9, 10 and 6 polynomials respectively. Let $\text{sat}(\mathbb{T}_i^*) = \mathbb{G}_i \cap \mathbf{K}[x_1, \dots, x_5]$ for $i = 1, 2, 3, 4$. We have

$$\text{sat}(\mathbb{T}_1^*) = \left[\begin{array}{l} 2x_3^2x_1 + x_2x_3^2 + x_3 + x_1, \\ 2x_3x_2^3x_1 + 4x_3x_2^2x_1^2 - x_4x_3x_2 + x_3x_2^2 + x_1x_4^2 - 2x_1x_4x_3 + x_1x_4x_2 + \\ 4x_1x_3x_2 + 2x_2^2x_1 + 4x_3x_1^2 - x_4 + x_2 + 2x_1, \\ x_4x_3x_2 - 2x_2^2x_1 + x_4^2x_3 - x_2 - 2x_1 + x_4, \\ -x_5^2x_1^2 + x_2x_5^2 - x_5 + x_4x_3 - x_2 - x_1, \\ x_5^2x_3 - x_3^2x_1 - x_3 - x_1 + x_5^2x_3^2x_1^2 + x_5x_3^2 + x_1x_5^2 - x_3^3x_4 + \\ 2x_5^2x_3^2x_1. \end{array} \right]$$

$$\text{sat}(\mathbb{T}_2^*) = \left[\begin{array}{l} -x_1 + x_2^2, \\ 2x_3^2x_1 + x_3 + x_1 + x_1^2x_3^2, \\ 2x_3x_2^3x_1 + 4x_3x_2^2x_1^2 - x_3x_2x_4 + x_3x_2^2 + x_4^2x_1 - 2x_4x_3x_1 + x_4x_2x_1 + \\ 4x_3x_2x_1 + 2x_1x_2^2 + 4x_3x_1^2 - x_4 + x_2 + 2x_1, \\ x_3x_2x_4 - 2x_1x_2^2 + x_3x_4^2 - x_2 - 2x_1 + x_4, \\ -x_5^2x_1^2 + x_2x_5^2 - x_5 + x_3x_4 - x_2 - x_1, \\ -x_3^2x_1 - x_3 - x_1 + x_5^2x_1 + x_1^2x_5^2x_3^2 + x_3^2x_5 - x_3^3x_4 + x_3x_5^2 \\ + 2x_5^2x_3^2x_1. \end{array} \right]$$

$$\text{sat}(\mathbb{T}_3^*) = \left[\begin{array}{l} 2x_1 + x_2, \\ x_3 + x_1, \\ 2x_3^2x_1 + x_3 + x_1 + x_1^2x_3^2, \\ x_5^2x_1^2 - x_1 + x_5 + 2x_5^2x_1 + x_4x_1, \\ x_5^2x_4x_1 + 2x_5^2x_4 + x_5x_4^2 - 2x_5x_4x_1 + 8x_5x_1^2 + x_4^2 - x_4, \\ 2x_4^2x_5^2 + x_4^3x_5 + 4x_5x_4x_1^2 + 16x_5x_1^3 + 9x_5^2x_4 + 4x_5x_4^2 + 32x_5x_1^2 - \\ 32x_5^2x_1 - 8x_5x_4x_1 + x_4^3 + 4x_4x_1^2 + 16x_1^3 - 4x_5x_4 + 8x_5x_1 - 6x_4 \\ + 3x_4^2 - 14x_4x_1 - 8x_1^2 - 16x_5 + 16x_1. \end{array} \right]$$

$$\text{sat}(\mathbb{T}_4^*) = \mathbb{T}_4^*.$$

It is easy to verify that $\text{Zero}(\text{sat}(\mathbb{T}_2^*))$, $\text{Zero}(\text{sat}(\mathbb{T}_3^*))$ and $\text{Zero}(\text{sat}(\mathbb{T}_4^*))$ are subvarieties of $\text{Zero}(\text{sat}(\mathbb{T}_1^*))$. Therefore, $\text{Zero}(\mathbb{P}) = \text{Zero}(\text{sat}(\mathbb{T}_1^*))$ is an unmixed decomposition.

By our improvement, \mathbb{P} is decomposition into two characteristic sets such that

$Zero(\mathbb{P}) = Zero(\mathbb{T}_1^*/x_3) \cup Zero(\mathbb{T}_4^*)$ since $\mathbb{U}_{\mathbb{T}_1^*} = \{x_3\}$ and $\mathbb{U}_{\mathbb{T}_4^*} = \emptyset$, where \mathbb{T}_1^* and \mathbb{T}_4^* as above. In order to determine $Zero(sa(\mathbb{T}_1^*))$, we only compute the *Gröbner* base \mathbb{G}_1 of $\mathbb{T}_1^* \cup \{1 - zx_3\}$ according to Theorem 4.1. The *Gröbner* base may be found to consist of 8 polynomials. Let $Ideal(\mathbb{T}_1^*) : x_3^\infty = \mathbb{G}_1 \cap \mathbf{K}[x_1, \dots, x_5]$. We have $Ideal(\mathbb{T}_1^*) : x_3^\infty = sat(\mathbb{T}_1^*)$, and remove \mathbb{T}_4^* according to *U3*, then we get the result as above.

Example 4.4. Let $\mathbb{P} = \{f_1, f_2, f_3, f_4\}$ be a polynomial set in $\mathbf{K}[x_1, x_2, x_3, x_4, x_5]$ where

$$\begin{aligned} f_1 &= 2x_2^2x_1 + x_1x_2 + x_5^2x_3 + 2x_5 + 2x_3, \\ f_2 &= x_3^2x_2 + x_3 + 2x_1x_2 + 3x_4^2x_1^2 + x_4^2x_2 + 2x_4, \\ f_3 &= x_3x_5^2 + 2x_5 + 2x_1x_2^2 + 3x_3 + x_3^2x_2 + 2x_1x_2 + x_2^3x_1, \\ f_4 &= 2x_5 + 3x_1x_2 + x_3x_5^2 + 4x_3 + 2x_2x_3^2. \end{aligned}$$

Under variable ordering $x_1 \prec x_2 \prec x_3 \prec x_4 \prec x_5$ \mathbb{P} is decomposed into eight characteristic sets \mathbb{T}_i such that

$$Zero(\mathbb{P}) = \bigcup_{i=1}^8 Zero(\mathbb{T}/ini(\mathbb{T}_i))$$

where

$$\begin{aligned} \mathbb{T}_1 &= [-x_2x_3^2 - x_3 + x_2^2x_1 - x_1x_2, -x_1x_4^2 - x_3x_4^2 - x_4 - x_3 + 2x_2 - x_1x_2 \\ &\quad - x_2^2x_1, x_3x_5^2 + 2x_5 + 2x_2^2x_1 + x_1x_2 + 2x_3], \\ \mathbb{T}_2 &= [x_1x_2^2 - x_1x_2, x_3, x_1x_4^2 + x_4 - 2x_2 + 2x_1x_2, 2x_5 + 3x_1x_2], \\ \mathbb{T}_3 &= [x_1, -x_3, -x_4 + 2x_2, 2x_5], \\ \mathbb{T}_4 &= [x_1x_2^2 - x_1x_2 - x_1^2x_2 + x_1, -x_3 - x_1, x_4 - 2x_2 + 2x_1x_2 - 2x_1 + x_1^2x_2, -x_1x_5^2 + \\ &\quad 2x_5 + 3x_1x_2 - 4x_1 + 2x_1^2x_2], \\ \mathbb{T}_5 &= [-x_1, -x_3, x_4 - 2x_2, 2x_5], \\ \mathbb{T}_6 &= [x_1, -x_3, x_4 - 2x_2, 2x_5], \\ \mathbb{T}_7 &= [-x_2, -x_3, -x_1x_4^2 - x_4, 2x_5], \\ \mathbb{T}_8 &= [-x_1, -x_2, -x_3, -x_4, 2x_5]. \end{aligned}$$

We can remove \mathbb{T}_8 according to Lemma 3.2, and remove $\mathbb{T}_i (i = 2, \dots, 7)$ by compute $sat(\mathbb{T}_i) (i = 2, \dots, 7)$ and *U3*, where

$$sat(\mathbb{T}_1) =$$

$$\left[\begin{array}{l} -x_2^2x_1 + x_3 + x_2x_3^2 + x_1x_2, \\ -2x_2^2x_3 + 4x_2^2x_1 - x_1x_2x_3 + x_3x_1x_2^2 + x_2x_3x_4 - x_1^2x_2^2 - x_1^2x_2^3 + \\ x_3x_1x_2^3 - x_4^2x_1^2x_2 + x_4^2x_1x_2^2 - x_4^2x_1x_2 - x_4x_1x_2 + x_1x_4^2 + x_4 - 2x_2, \\ x_1x_4^2 + x_4^2x_3 + x_4 + x_3 - 2x_2 + x_1x_2 + x_2^2x_1, \\ 2x_5 + x_5^2x_2^2x_1 + 2x_3x_1x_2^3 - x_5^2x_1x_2 + x_3x_1x_2^2 + 2x_2x_3x_5 + 4x_2^2x_1 \\ - x_1x_2, \\ 2x_2^2x_1 + x_1x_2 + x_5^2x_3 + 2x_5 + 2x_3, \\ -2x_4^2x_5 - x_2x_4 + 2x_2^2 - 8x_2^3 - 2x_2^3x_3 + 4x_4x_2^2 - 4x_2^4x_3 - 8x_5x_2^2 \\ - 2x_5x_2^3x_3 - 2x_5x_2^2x_3 - 2x_5^2x_2^3 + 2x_5^2x_2^2 + x_5^2x_4x_2^2 + 2x_4x_2^3x_3 \\ - x_5^2x_4x_2 + x_2^2x_4x_3 - 2x_5x_2^2x_4^2 + 2x_4^2x_1x_5x_2 + 2x_5x_4x_2 + 2x_2x_3x_5 \\ + 2x_5x_1x_2^2 + 2x_5x_1x_2^3 + 2x_5x_4^2x_2, \\ -8x_2 + 4x_4 - 4x_5 + 2x_1x_2 + 4x_1x_4^2 + x_5^2x_4 - 4x_2^2x_3 - 2x_4^2x_5 + \\ 4x_2^2x_1 - 2x_1x_2x_3 + x_3x_1x_2^2 + 2x_2x_3x_4 - 2x_1^2x_2^2 - 2x_1^2x_2^3 + x_4^2x_5^2x_1 \\ - 2x_2x_3x_5 - 2x_4^2x_1^2x_2 - 3x_4^2x_1x_2 - 2x_4x_1x_2 + 2x_5^2x_1x_2 - 2x_5^2x_2. \end{array} \right]$$

then $\text{Zero}(\mathbb{P}) = \text{Zero}(\text{sat}(\mathbb{T}_1))$.

By our improvement we get $\text{CharserA}(\mathbb{P}) = \{\mathbb{T}^*\}$, where

$$\mathbb{T}^* = \left[\begin{array}{l} -x_2x_3^2 - x_3 + x_1x_2^2 - x_1x_2, \\ x_1x_4^2 + x_3x_4^2 + x_4 + x_3 - 2x_2 + x_1x_2 + x_1x_2^2, \\ x_3x_5^2 + 2x_5 + 2x_1x_2^2 + x_1x_2 + 2x_3. \end{array} \right]$$

It is easy to see that $\mathbb{U}_{\mathbb{T}^*} = \emptyset$, then $\text{Zero}(\text{sat}(\mathbb{T}^*)) = \text{Zero}(\mathbb{T}^*)$ according to Lemma 4.1, so we get $\text{Zero}(\mathbb{P}) = \text{Zero}(\mathbb{T}^*)$ directly.

References

- [1] Buchberger, B. Ein Algorithmus Zum Auffinden der Basiselement des Restklassenringes nach einem nulldimensionalen Polynom. Ph.D. thesis, Univesitat Innsbruck, Austria, 1965.
- [2] Buchberger, B. Groebner bases: An gorithmic method in polynomial ideal theory. In: *multidimensional systems Theory* (Bose, N.K, ed.), Reidel, Dorderecht, 1985:184-232.
- [3] Chou, S.-C., Gao, X.-S. Ritt-Wu's decomposition algorithm and geometry theorem proving. In: *Proceeding CADE-10*, Kaiserslautern, July 24-27,1990. Springer, Berlin Heidelberg New York, pp.207-220(Lecture notes in computer science, vol.449)[also as Tech. Rep. TR-89-09, Department of Computer science, The University of Texas at Austin, USA].
- [4] Chou, S.-C., Gao, X.-S. Solving parametric algebraic systems. In *Proceedings IS-SAAC'92*,1992, 335-341.
- [5] Chou, S.-C., Schelter, W.F., Yang, J.-G. An algorithm for constructing Gröbner bases from characteristic sets and its application to geometry. *Algorithmica* **5**(1990):147-154.

- [6] Gao, X.-S., Chou, S.-C. On the dimension of an arbitrary ascending chain. *Chinese Sci. Bull.***38**(1993):199-804.
- [7] Gao, X.-S., Wang, D.-K., Liao, Q. and Yang, H. Equation solving and Maciuen Proving-Problem Solving with MMP. Science Press, Beijing, 2006 (in Chinese).
- [8] Gianni, P., Trager, B. M., Zacharias, G. Gröbner bases and primary decomposition of polynomial ideals. *J. Symb. Comput.***6**(1988):149-167.
- [9] Huang Fangjian. Researches on Algorithms of Decomposition of Polynomial System. Ph.D. Chengdu Institute of Computer Applications, China, 2007.
- [10] Kalkbrener, M. A generalized Euclidean algorithm for computing triangular representations of algebraic varieties. *J. Symb. Comput.*, 1993, **15**:143-167.
- [11] Lazard, D. A new method for solving algebraic systems of positive dimension. *Discrete Appl. math.*, 1991, **33**:147-160.
- [12] Li, Y.-B. An alternative algorithm for computing the pseudo-remainder of multivariate polynomials. *Applied Math Comput.*, 2006, **173**:484-492.
- [13] Li, Y.-B. Some properties of triangular sets and improvement upon algorithm CharSer. In J. Calmet, T. Ida, and D. Wang eds, AISC2006, LNAI 4120. Springer-Verlag, Berlin/Heidelberg, 2006 82-93.
- [14] Li, Y.-B. Applications of the theory of weakly nondegenerate conditions to zero decomposition for polynomial systems. *J. Symb. Comput.*, 2004, **38**:818-832.
- [15] Mishra, B. Algorithmic algebra. Springer, Berlin Heidelberg New York Tokyo (Texts and monographs in computer science), 1993.
- [16] Moreno Maza, M. On triangular decompositions of algebraic varieties. *In Proceedings of MEGA 2000*. 2000. .
- [17] Wang, D. An elimination method for polynomial systems. *J. Symb. Comput.*, 1993, **16**:83-114.
- [18] Wang, D. Elimination methods. Springer, Wien/New York, 2001.
- [19] Wang, D. Computing triangular systems and regular systems. *J. Symb. Comput.*, 2000, **30**:221-236.
- [20] Wang, D. Characteristic sets and zero structure of polynomial sets. Lecture Notes, RISC-Linz, Johannes Kepler University, Austria (1989-1995) [also available from <http://calfor.lip6.fr/~wang/manu.html>].
- [21] Wang, D. An implementation method of the characteristic set method in Maple. In: Pfalzgraf, J., Wang, D (eds.): Automated practical reasoning: Algebraic approaches. Springer, Wien New York, pp. 187-201 (1995).

- [22] Wu, W.-T. On the decision problem and the meahanization of theorem-proving in elementary geometry. *Scientia sinica*, 1978, **21**:159-172.
- [23] Wu, W.-T. On zeros of algebraic equations-An application o f Ritt principle. *kerue Tongbao*, 1986. **31**:1-5.
- [24] Wu, W.-T. A zero structure theorem for polynomial equations solving. MM research Preprints, 1987. **1**:2-12.
- [25] Yang, L., Zhang, J.-Z. Search dependency between algebraic equation: An algorithm applied to autaoated reasoning. *Technical Report ICTP/91/6*, International Center For Theoretical Physics, International Atomic Energy Agency, Miramare, Trieste, 1991.
- [26] Yang, L., Zhang J.-Z., Hou, X.-R. non-linear Equation system and Automated Theorem Proving. Shanghai Sci. Tech. Education Publ. House, Shanghai, 1996 (in Chinese).
- [27] Yang, L., Hou, X.-R. Gather-and-Shift. A symbolic method for solving polnomials. In: *Proceedings ATCM'95*, Singapore, 1995, 771-780.
- [28] Zhang, J.-Z., Yang, L., Hou, X.-R. A note on Wu Wen-Tsün's nondegenerate condition. *Technical Report ICTP/91/160*, International Center for Theoretical Physics, International Atomic Energy Agency, Miramare, Trieste, 1991; Also in Chises Science Bullentin, 1993, **38**:1, 86-87.